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1st Edition, 2nd Printing

by

Johnny Li, Ph.D., FSA

Andrew Ng, Ph.D., FSA

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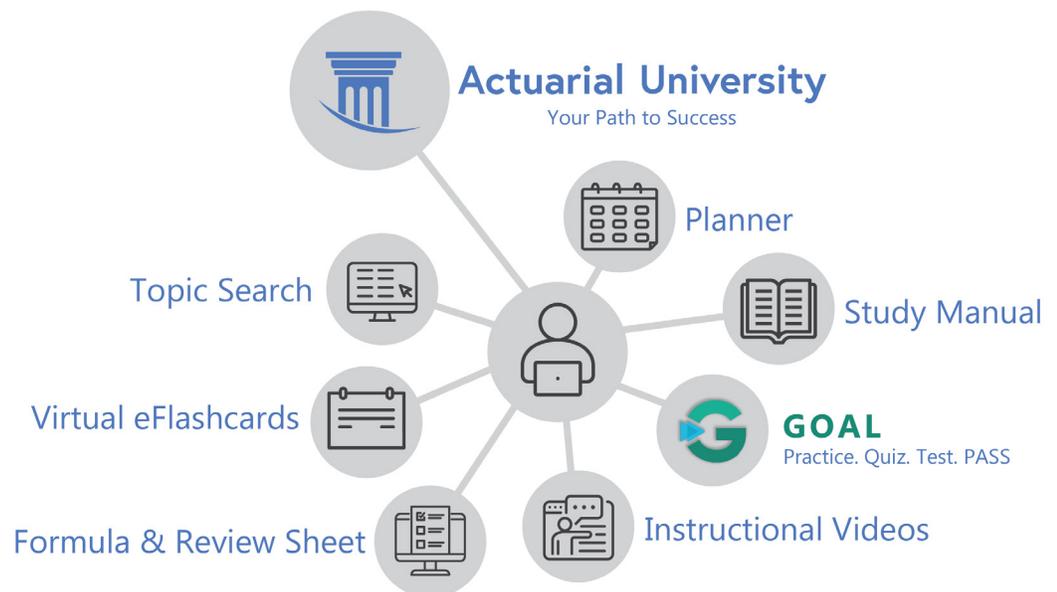
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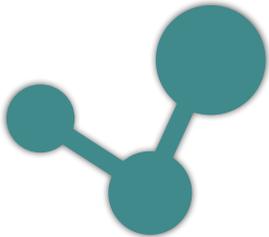
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Pareto Distribution ▾

The (Type II) Pareto distribution with parameters $\alpha, \beta > 0$ has pdf

$$f(x) = \frac{\alpha\beta^{\alpha}}{(x+\beta)^{\alpha+1}}, \quad x > 0$$

and cdf

$$F_P(x) = 1 - \left(\frac{\beta}{x+\beta}\right)^{\alpha}, \quad x > 0.$$

If X is Type II Pareto with parameters α, β, then

$$E[X] = \frac{\beta}{\alpha - 1} \text{ if } \alpha > 1,$$

and

$$Var[X] = \frac{\alpha\beta^2}{\alpha - 2} - \left(\frac{\alpha\beta}{\alpha - 1}\right)^2 \text{ if } \alpha > 2.$$

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Question   Difficulty: Mastery ⓘ

At time $t = 0$ year, Donald puts \$1,000 into a fund crediting interest at a nominal rate of i compounded semiannually.

At time $t = 2$ years, Lewis puts \$1,000 into a different fund crediting interest at a force $\delta_t = 1/(5 + t)$ for all t.

At time $t = 16$ years, the amounts in each fund will be equal.

Calculate i.

Flag problems for review, record notes, and email your professor.

View difficulty level.

Possible Answers

| ✕ 6.9% | ✓ 7.0% | C 7.1% | B 7.2% | A 7.3% |

Help Me Start   ^

Equate the expressions for the AVs at $t = 16$. Then solve for $i^{(2)}$:

Helpful strategies to get you started.

Solution   ^

Equate the expressions for for the AVs at $t = 16$ and calculate $i^{(2)}$:

$(1 + i^{(2)}/2)^{32} = 3$

$(1 + i^{(2)}/2) = 3^{(1/32)} = 1.03493$

▶ 0:08 / 0:59 🔊 ⋮

$i^{(2)}/2 = 0.03493$

Donald: $a(16) = (1 + i^{(2)}/2)^{2 \cdot 16} = (1 + i^{(2)}/2)^{32}$

$i^{(2)} = 7.0\%$

Lewis: $a(16) = e^{\int_2^{16} \frac{1}{5+t} dt} = e^{\ln(5+t)|_2^{16}} = e^{\ln(21) - \ln(7)} = 21/7 = 3$

Full solutions, with detailed explanations to deepen your understanding.

Common Questions & Errors   ^

Student Question 1: After solving this problem I got .069855. Are we expected to round to .07?

Answer: The provided answer choices are all rounded to 1 decimal place. So the answer 6.9855% should be rounded to 7.0% to be correct to 1 decimal place.

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General Probability	Random Variables with Univariate Probability Distributions	Random Variables with Univariate Probability Distributions
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General Probability — Set functions including set notation and basic elements of probability

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Advanced	60	169 / 304
Mastery	78	43 / 304

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05/24/2022 10:34:05	05/24/2022 14:57:49	05/24/2022 14:57:49	Practice Session	Addition and ...	80	Complete	Review
05/21/2022 19:32:50	05/23/2022 20:02:00		Simulated Exam	Exam 2	30	Reviewing	Complete
05/17/2022 15:19:19	05/17/2022 15:46:03	05/23/2022 14:15:29	Simulated Exam	Exam 6	30	Complete	Review
05/14/2022 11:26:59	05/14/2020 12:02:36	05/23/2022 11:57:47	Quiz	Conditional D...	20	Complete	Review

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Contents

Chapter 2

Life Tables

Objectives

1. To apply life tables

2. To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality

3. To calculate moments for future lifetime random variables

4. To understand and model the effect of selection

Actuaries use spreadsheets extensively in practice. It would be very helpful if we could express survival distributions in a tabular form. Such tables, which are known as life tables, are the focus of this chapter.

2.1 Life Table Functions

Below is an excerpt of a (hypothetical) life table. In what follows, we are going to define the functions l_x and d_x, and explain how they are applied.

x	l_x	d_x
0	1000	16
1	984	7
2	977	12
3	965	75
4	890	144

In this hypothetical life table, the value of l_0 is 1,000. This starting value is called the radix of the life table. For $x = 1, 2, \ldots$, the function l_x stands for the expected number of persons who can survive to age x. Given an assumed value of l_0, we can express any survival function $S_0(x)$ in a tabular form by using the relation

$$l_x = l_0 S_0(x).$$

In the other way around, given the life table function l_x, we can easily obtain values of $S_0(x)$ for integral values of x using the relation

$$S_0(x) = \frac{l_x}{l_0}.$$

Furthermore, we have

$$_tp_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{l_{x+t}/l_0}{l_x/l_0} = \frac{l_{x+t}}{l_x},$$

which means that we can calculate $_tp_x$ for all integral values of t and x from the life table function l_x.

The difference $l_x - l_{x+t}$ is the expected number of deaths over the age interval of $[x, x+t)$. We denote this by $_td_x$. It immediately follows that $_td_x = l_x - l_{x+t}$.

We can then calculate $_tq_x$ and $_{m|n}q_x$ by the following two relations:

$$_tq_x = \frac{_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}, \qquad _{m|n}q_x = \frac{_nd_{x+m}}{l_x} = \frac{l_{x+m} - l_{x+m+n}}{l_x}.$$

When $t = 1$, we can omit the subscript t and write $_1d_x$ as d_x. By definition, we have

$$_td_x = d_x + d_{x+1} + \cdots + d_{x+t-1}.$$

Graphically,
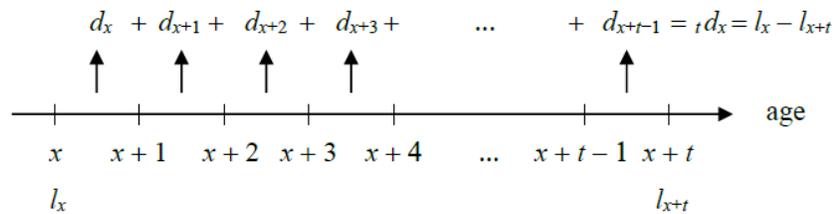
Also, when $t = 1$, we have the following relations:

$$d_x = l_x - l_{x+1}, \ \ p_x = \frac{l_{x+1}}{l_x}, \ \text{and} \ q_x = \frac{d_x}{l_x}.$$

Summing up, with the life table functions l_x and d_x, we can recover survival probabilities $_tp_x$ and death probabilities $_tq_x$ for all integral values of t and x easily.

Life Table Functions

(2.1)
$$_tp_x = \frac{l_{x+t}}{l_x}$$

(2.2)
$$_td_x = l_x - l_{x+t} = d_x + d_{x+1} + \cdots + d_{x+t-1}$$

(2.3)
$$_tq_x = \frac{_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}$$

Some FAM-L exam and LTAM (the predecessor of FAM-L) questions are based on the Standard Ultimate Life Table, and some MLC (the predecessor of LTAM) exam questions are based on the Illustrative Life Table. The Standard Ultimate Life Table can be found at the SOA's website:

https://www.soa.org/Education/Exam-Req/edu-exam-ltam-detail.aspx,

and the Illustrative Life Table is provided in Appendix 2 of this study manual. The two tables have very similar formats. They contain a lot of information. For now, you only need to know and use the first three columns: x, l_x, and q_x (Standard Ultimate Life Table) and $1000q_x$ (Illustrative Life Table). For example, to obtain q_{43}, simply use the column labeled q_x. You should obtain $q_{43} = 0.000656$ (from the Standard Ultimate Life Table). It is also possible, but more tedious, to calculate q_{43} using the column labeled l_x; we have $q_{43} = 1 - l_{44}/l_{43} = 1 - 99104.3/99169.4 = 0.000656452$.

To get values of $_tp_x$ and $_tq_x$ for $t > 1$, you should always use the column labeled l_x. For example, we have $_5p_{61} = l_{66}/l_{61} = 94020.3/96305.8 = 0.976268$ and $_5q_{61} = 1 - _5p_{61} = 1 - 0.976268 = 0.023732$ (from the Standard Ultimate Life Table). Here, you should not base your calculations on the column labeled q_x, partly because that would be a lot more tedious, and partly because that may lead to a huge rounding error.

Example 2.1.

You are given the following excerpt of a life table:

x	l_x	d_x
20	96178.01	99.0569
21	96078.95	102.0149
22	95976.93	105.2582
23	95871.68	108.8135
24	95762.86	112.7102
25	95650.15	116.9802

Calculate the following:

(a) $_5p_{20}$

(b) q_{24}

(c) $_{4|1}q_{20}$

Solution:

(a) $_5p_{20} = \dfrac{l_{25}}{l_{20}} = \dfrac{95650.15}{96178.01} = 0.994512.$

(b) $q_{24} = \dfrac{d_{24}}{l_{24}} = \dfrac{112.7102}{95762.86} = 0.001177.$

(c) $_{4|1}q_{20} = \dfrac{_1d_{24}}{l_{20}} = \dfrac{112.7102}{96178.01} = 0.001172.$

$\square$

Example 2.2. ⚛

You are given:

  (i) $S_0(x) = 1 - \dfrac{x}{100}, \quad 0 \le x \le 100$

 (ii) $l_0 = 100$

 (a) Find an expression for l_x for $0 \le x \le 100$.

 (b) Calculate q_2.

 (c) Calculate $_3q_2$.

Solution:

 (a) $l_x = l_0 S_0(x) = 100 - x.$

 (b) $q_2 = \dfrac{l_2 - l_3}{l_2} = \dfrac{98 - 97}{98} = \dfrac{1}{98}.$

 (c) $_3q_2 = \dfrac{l_2 - l_5}{l_2} = \dfrac{98 - 95}{98} = \dfrac{3}{98}.$

                                                   □

⚛ In Exam FAM-L, you may need to deal with a mixture of two populations. As illustrated in the following example, the calculation is a lot more tedious when two populations are involved.

Example 2.3. ⚛

For a certain population of 20 year olds, you are given:

  (i) $2/3$ of the population are nonsmokers, and $1/3$ of the population are smokers.

 (ii) The future lifetime of a nonsmoker is uniformly distributed over $[0, 80)$.

(iii) The future lifetime of a smoker is uniformly distributed over $[0, 50)$.

Calculate $_5p_{40}$ for a life randomly selected from those surviving to age 40.

Solution: The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 20.

- Suppose that there are l_{20} persons in the entire population initially. At time 0 (i.e., at age 20), there are $\dfrac{2}{3} l_{20}$ nonsmokers and $\dfrac{1}{3} l_{20}$ smokers.

- For nonsmokers, the proportion of individuals who can survive to age 40 is $1 - 20/80 = 3/4$. For smokers, the proportion of individuals who can survive to age 40 is $1 - 20/50 = 3/5$. At age 40, there are $\dfrac{3}{4} \dfrac{2}{3} l_{20} = 0.5 l_{20}$ nonsmokers and $\dfrac{3}{5} \dfrac{1}{3} l_{20} = 0.2 l_{20}$ smokers. Hence, among those who can survive to age 40, $5/7$ are nonsmokers and $2/7$ are smokers.

Second, we need to calculate the probabilities of surviving from age 40 to age 45 for both smokers and nonsmokers.

- For a nonsmoker at age 40, the remaining lifetime is uniformly distributed over $[0, 60)$. This means that the probability for a nonsmoker to survive from age 40 to age 45 is $1 - 5/60 = 11/12$.

- For a smoker at age 40, the remaining lifetime is uniformly distributed over $[0, 30)$. This means that the probability for a smoker to survive from age 40 to age 45 is $1 - 5/30 = 5/6$.

Finally, for the whole population, we have

$$_5p_{40} = \frac{5}{7} \times \frac{11}{12} + \frac{2}{7} \times \frac{5}{6} = \frac{25}{28}.$$

$\square$

2.2 Fractional Age Assumptions

We have demonstrated that given a life table, we can calculate values of $_tp_x$ and $_tq_x$ when both t and x are integers. But what if t and/or x are not integers? In this case, we need to make an assumption about how the survival function behaves between two integral ages. We call such an assumption a fractional age assumption.

In Exam FAM-L, you are required to know two fractional age assumptions:

1. Uniform distribution of death

2. Constant force of mortality

We go through these assumptions one by one.

Assumption 1: Uniform Distribution of Death

The Uniform Distribution of Death (UDD) assumption is extensively used in the Exam FAM-L syllabus. The idea behind this assumption is that we use a bridge, denoted by U, to connect the (continuous) future lifetime random variable T_x and the (discrete) curtate future lifetime random variable K_x. The idea is illustrated diagrammatically as follows:
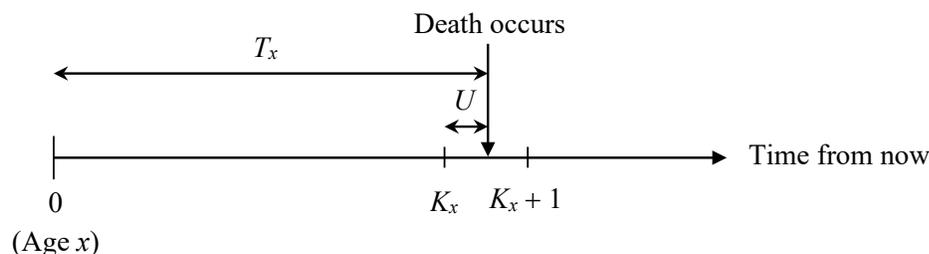
It is assumed that U follows a uniform distribution over the interval $[0, 1]$, and that U and K_x are independent. Then, for $0 \le r < 1$ and an integral value of x, we have

$$
\begin{aligned}
_r q_x &= \Pr(T_x \le r) \\
&= \Pr(U < r \cap K_x = 0) \\
&= \Pr(U < r)\Pr(K_x = 0) \\
&= r q_x.
\end{aligned}
$$

The second last step follows from the assumption that U and K_x are independent, while the last step follows from the fact that U follows a uniform distribution over $[0, 1]$.

Key Equation for the UDD Assumption

(2.4) $\qquad\qquad\qquad\qquad {}_r q_x = r q_x, \quad \text{for } 0 \le r < 1$

This means that under UDD, we have, for example, ${}_{0.4}q_{50} = 0.4 q_{50}$. The value of q_{50} can be obtained straightforwardly from the life table. To calculate ${}_r p_x$, for $0 \le r < 1$, we use ${}_r p_x = 1 - {}_r q_x = 1 - r q_x$. For example, we have ${}_{0.1}p_{20} = 1 - 0.1 q_{20}$.

Equation (2.4) is equivalent to a linear interpolation between l_x and l_{x+1}, that is,

$$
l_{x+r} = (1 - r)l_x + r l_{x+1}.
$$

Proof:

$$
\begin{aligned}
_r p_x &= 1 - r q_x = (1 - r) + r p_x \\
\frac{l_{x+r}}{l_x} &= (1 - r) + r \frac{l_{x+1}}{l_x} \\
l_{x+r} &= (1 - r)l_x + r l_{x+1}
\end{aligned}
$$

$\square$

You will find this equation – the interpolation between l_x and l_{x+1} – very useful if you are given a table of l_x (instead of q_x).

Application of the UDD Assumption to l_x

(2.5) $\qquad\qquad\quad l_{x+r} = (1 - r)l_x + r l_{x+1}, \quad \text{for } 0 \le r < 1$

What if the subscript on the left-hand-side of ${}_r q_x$ is greater than 1? In this case, we should first use equation (1.6) from Chapter 1 to break down the probability into smaller portions. As an example, we can calculate ${}_{2.5}p_{30}$ as follows:

$$
_{2.5}p_{30} = {}_2 p_{30} \times {}_{0.5}p_{32} = {}_2 p_{30} \times (1 - 0.5 q_{32}).
$$

The value of ${}_2 p_{30}$ and q_{32} can be obtained from the life table straightforwardly.

What if the subscript on the right-hand-side is not an integer? In this case, we should make use of a special trick, which we now demonstrate. Let us consider ${}_{0.1}p_{5.7}$ (both subscripts are not integers). The trick is that we multiply this probability with ${}_{0.7}p_5$, that is,

$$
_{0.7}p_5 \times {}_{0.1}p_{5.7} = {}_{0.8}p_5.
$$

This gives $_{0.1}p_{5.7} = \dfrac{_{0.8}p_5}{_{0.7}p_5} = \dfrac{1 - 0.8q_5}{1 - 0.7q_5}$. The value of q_5 can be obtained from the life table.

To further illustrate this trick, let us consider $_{3.5}p_{4.6}$: This probability can be evaluated from the following equation:

$$_{0.6}p_4 \times {_{3.5}p_{4.6}} = {_{4.1}p_4}.$$

Then, we have $_{3.5}p_{4.6} = \dfrac{_{4.1}p_4}{_{0.6}p_4} = \dfrac{_{4}p_4\,_{0.1}p_8}{_{0.6}p_4} = \dfrac{_{4}p_4(1 - 0.1q_8)}{1 - 0.6q_4}$, and finally $_{3.5}q_{4.6} = 1 - \dfrac{_{4}p_4(1 - 0.1q_8)}{1 - 0.6q_4}$.

The values of $_{4}p_4$, q_8 and q_4 can be obtained from the life table.

Let us study the following example.

Example 2.4. You are given the following excerpt of a life table:

x	l_x	d_x
60	100000	300
61	99700	400
62	99300	500
63	98800	600
64	98200	700
65	97500	800

Assuming uniform distribution of deaths between integral ages, calculate the following:

(a) $_{0.26}p_{61}$

(b) $_{2.2}q_{60}$

(c) $_{0.3}q_{62.8}$

Solution:

(a) $_{0.26}p_{61} = 1 - {_{0.26}q_{61}} = 1 - 0.26 \times 400/99700 = 0.998957.$

Alternatively, we can calculate the answer by using a linear interpolation between l_{61} and l_{62} as follows:

$$l_{61.26} = (1 - 0.26)l_{61} + 0.26l_{62} = 0.74 \times 99700 + 0.26 \times 99300 = 99596.$$

It follows that $_{0.26}p_{61} = l_{61.26}/l_{61} = 99596/99700 = 0.998957.$

(b) $_{2.2}q_{60} = 1 - {_{2.2}p_{60}} = 1 - {_{2}p_{60}} \times {_{0.2}p_{62}} = 1 - {_{2}p_{60}} \times (1 - 0.2q_{62})$

$$= 1 - \frac{l_{62}}{l_{60}}\left(1 - 0.2 \times \frac{d_{62}}{l_{62}}\right) = 1 - \frac{99300}{100000}\left(1 - 0.2 \times \frac{500}{99300}\right) = 0.008.$$

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} as follows:

$$l_{62.2} = (1 - 0.2)l_{62} + 0.2l_{63} = 0.8 \times 99300 + 0.2 \times 98800 = 99200.$$

It follows that $_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 1 - 99200/100000 = 0.008.$

(c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute $_{0.3}p_{62.8}$ from the following equation:

$$_{0.8}p_{62} \times {}_{0.3}p_{62.8} = {}_{1.1}p_{62}.$$

Trick
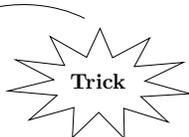
Rearranging the equation above, we have

$$_{0.3}p_{62.8} = \frac{_{1.1}p_{62}}{_{0.8}p_{62}} = \frac{p_{62}\, {}_{0.1}p_{63}}{_{0.8}p_{62}} = \frac{p_{62}\,(1 - 0.1q_{63})}{1 - 0.8q_{62}} = \frac{\dfrac{98800}{99300}\left(1 - 0.1 \times \dfrac{600}{98800}\right)}{1 - 0.8 \times \dfrac{500}{99300}}$$

$$= 0.998382.$$

Hence, $_{0.3}q_{62.8} = 1 - 0.998382 = 0.001618$.

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} and another interpolation between l_{63} and l_{64}:

First,

$$l_{62.8} = (1 - 0.8)l_{62} + 0.8l_{63} = 0.2 \times 99300 + 0.8 \times 98800 = 98900.$$

Second,

$$l_{63.1} = (1 - 0.1)l_{63} + 0.1l_{64} = 0.9 \times 98800 + 0.1 \times 98200 = 98740.$$

Finally,

$$_{0.3}q_{62.8} = 1 - {}_{0.3}p_{62.8} = 1 - l_{63.1}/l_{62.8} = 1 - 98740/98900 = 0.001618.$$

$\square$

Sometimes, you may be asked to calculate the density function of T_x and the force of mortality from a life table. Under UDD, we have the following equation for calculating the density function:

$$f_x(r) = q_x, \quad 0 \le r < 1.$$

Proof: $f_x(r) = \dfrac{\mathrm{d}}{\mathrm{d}r}F_x(r) = \dfrac{\mathrm{d}}{\mathrm{d}r}\mathrm{Pr}(T_x \le r) = \dfrac{\mathrm{d}}{\mathrm{d}r}{}_rq_x = \dfrac{\mathrm{d}}{\mathrm{d}r}(rq_x) = q_x.$ $\square$

Under UDD, we have the following equation for calculating the force of mortality:

$$\mu_{x+r} = \frac{q_x}{1 - rq_x}, \quad 0 \le r < 1.$$

Proof: In general, $f_x(r) = {}_rp_x\mu_{x+r}$. Under UDD, we have $f_x(r) = q_x$ and $_rp_x = 1 - rq_x$. The result follows. $\square$

Let us take a look at the following example.

Example 2.5. 🌀 **[Course 3 Spring 2000 #12]**

For a certain mortality table, you are given:

  (i) $\mu_{80.5} = 0.0202$

  (ii) $\mu_{81.5} = 0.0408$

  (iii) $\mu_{82.5} = 0.0619$

  (iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782   (B) 0.0785   (C) 0.0790   (D) 0.0796   (E) 0.0800

Solution: The probability that a person age 80.5 will die within two years is $_2q_{80.5}$. We have

$$_{0.5}p_{80} \times {_2}p_{80.5} = {_{2.5}}p_{80}.$$

This gives

$$_2p_{80.5} = \frac{_2p_{80}\ _{0.5}p_{82}}{_{0.5}p_{80}} = \frac{p_{80}p_{81}\left(1 - 0.5q_{82}\right)}{1 - 0.5q_{80}} = \frac{\left(1 - q_{80}\right)\left(1 - q_{81}\right)\left(1 - 0.5q_{82}\right)}{1 - 0.5q_{80}}.$$

We then need to find q_{80}, q_{81} and q_{82} from the information given in the question. Using $\mu_{80.5}$, we have $\mu_{80.5} = \dfrac{q_{80}}{1 - 0.5q_{80}} \Rightarrow q_{80} = 0.0200$. Similarly, by using $\mu_{81.5}$ and $\mu_{82.5}$, we obtain $q_{81} = 0.0400$ and $q_{82} = 0.0600$.

Substituting q_{80}, q_{81} and q_{82}, we obtain $_2p_{80.5} = 0.921794$, and hence $_2q_{80.5} = 1 - {_2}p_{80.5} = 0.0782$. Hence, the answer is (A).

$\square$

Assumption 2: Constant Force of Mortality

The idea behind this assumption is that for every age x, we approximate μ_{x+k} for $0 \le r < 1$ by a constant, which we denote by $\tilde{\mu}_x$. This means

$$\int_0^1 \mu_{x+u}\ \mathrm{d}u = \int_0^1 \tilde{\mu}_x\ \mathrm{d}u = \tilde{\mu}_x,$$

which implies $p_x = e^{-\tilde{\mu}_x}$ and $\tilde{\mu}_x = -\ln\left(p_x\right)$.

We are now ready to develop equations for calculating various death and survival probabilities. First of all, for any integer-valued x, we have

$$_rp_x = \left(p_x\right)^r, \quad 0 \le r < 1.$$

Proof: $_rp_x = \exp\left(-\int_0^r \mu_{x+u}\mathrm{d}u\right) = \exp\left(-\int_0^r \tilde{\mu}_x\mathrm{d}u\right) = e^{-\tilde{\mu}_x r} = (e^{-\tilde{\mu}_x})^r = (p_x)^r.$ $\square$

For example, $_{0.3}p_{50} = \left(p_{50}\right)^{0.3}$, and $_{0.4}q_{62} = 1 - {_{0.4}}p_{62} = 1 - \left(p_{62}\right)^{0.4}$. We can generalize the equation above to obtain the following key formula.

> **Key Equation for the Constant Force of Mortality Assumption**
>
> **(2.6)** $_r p_{x+u} = (p_x)^r, \quad \text{for } 0 \le r < 1 \text{ and } r + u \le 1$

Proof: $_r p_{x+u} = \exp\left(-\int_0^r \mu_{x+u+t} \mathrm{d}t\right) = \exp\left(-\int_0^r \tilde{\mu}_x \mathrm{d}t\right) = e^{-\tilde{\mu}_x r} = (p_x)^r.$

[The second step follows from the fact that given $0 \le r < 1$, $u + t$ is always less than or equal to 1 when $0 \le t \le r$.] □

Notice that the key equation for the constant force of mortality assumption is based on p, while that for the UDD assumption is based on q.

This key equation means that, for example, $_{0.2}p_{30.3} = (p_{30})^{0.2}$. Note that the subscript u on the right-hand-side does not appear in the result, provided that the condition $r + u \le 1$ is satisfied. But what if $r + u > 1$? The answer is very simple: Split the probability! To illustrate, let us consider $_{0.8}p_{30.3}$. (Here, $r + u = 0.8 + 0.3 = 1.1 > 1$.) By using equation (1.6) from Chapter 1, we can split $_{0.8}p_{30.3}$ into two parts as follows:

$$_{0.8}p_{30.3} = {}_{0.7}p_{30.3} \times {}_{0.1}p_{31}.$$

We intentionally consider a duration of 0.7 years for the first part, because $0.3 + 0.7 = 1$, which means we can apply the key equation $_r p_{x+u} = (p_x)^r$ to it. As a result, we have

$$_{0.8}p_{30.3} = (p_{30})^{0.7} \times (p_{31})^{0.1}.$$

The values of p_{30} and p_{31} can be obtained from the life table straightforwardly.

To further illustrate, let us consider $_{5.6}p_{40.8}$. We can split it as follows:

$$_{5.6}p_{40.8} = {}_{0.2}p_{40.8} \times {}_{5.4}p_{41} = {}_{0.2}p_{40.8} \times {}_5 p_{41} \times {}_{0.4}p_{46} = (p_{40})^{0.2} \times {}_5 p_{41} \times (p_{46})^{0.4}.$$

The values of p_{40}, $_5 p_{41}$ and p_{46} can be obtained from the life table straightforwardly.

Interestingly, equation (2.6) implies that for $0 \le r < 1$, the value of $\ln(l_{x+r})$ can be obtained by a linear interpolation between the values of $\ln(l_x)$ and $\ln(l_{x+1})$.

Proof: Setting $u = 0$ in equation (2.6), we have

$$_r p_x = (p_x)^r$$
$$\frac{l_{x+r}}{l_x} = \left(\frac{l_{x+1}}{l_x}\right)^r$$
$$\ln(l_{x+r}) - \ln(l_x) = r\ln(l_{x+1}) - r\ln(l_x)$$
$$\ln(l_{x+r}) = (1-r)\ln(l_x) + r\ln(l_{x+1})$$

□

You will find this equation – the interpolation between $\ln(l_x)$ and $\ln(l_{x+1})$ – useful when you are given a table of l_x.

<div style="border:1px solid blue">

Application of the Constant Force of Mortality Assumption to l_x

$$\ln(l_{x+r}) = (1-r)\ln(l_x) + r\ln(l_{x+1}), \quad \text{for } 0 \le r < 1$$

</div>

Example 2.6.

Assuming constant force of mortality between integral ages, repeat Example 2.4.

Solution:

(a) $_{0.26}p_{61} = (p_{61})^{0.26} = (99300/99700)^{0.26} = 0.998955.$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{61})$ and $\ln(l_{62})$ as follows: $\ln(l_{61.26}) = (1-0.26)\ln(l_{61}) + 0.26\ln(l_{62})$, which gives $l_{61.26} = 99595.84526$. Hence, $_{0.26}p_{61} = l_{61.26}/l_{61} = 99595.84526/99700 = 0.998955.$

(b) $_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_2p_{60} \times {}_{0.2}p_{62} = 1 - {}_2p_{60} \times (p_{62})^{0.2}$

$$= 1 - \frac{l_{62}}{l_{60}}\left(\frac{l_{63}}{l_{62}}\right)^{0.2} = 1 - \frac{99300}{100000}\left(\frac{98800}{99300}\right)^{0.2} = 0.008002.$$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{62})$ and $\ln(l_{63})$ as follows: $\ln(l_{62.2}) = (1-0.2)\ln(l_{62}) + 0.2\ln(l_{63})$, which gives $l_{62.2} = 99199.79798$. Hence, $_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 0.008002.$

(c) First, we consider $_{0.3}p_{62.8}$:

$$_{0.3}p_{62.8} = {}_{0.2}p_{62.8} \times {}_{0.1}p_{63} = (p_{62})^{0.2}(p_{63})^{0.1}.$$

Hence,

$$_{0.3}q_{62.8} = 1 - (p_{62})^{0.2}(p_{63})^{0.1} = 1 - \left(\frac{l_{63}}{l_{62}}\right)^{0.2}\left(\frac{l_{64}}{l_{63}}\right)^{0.1}$$

$$= 1 - \left(\frac{98800}{99300}\right)^{0.2}\left(\frac{98200}{98800}\right)^{0.1} = 0.001617.$$

Alternatively, we can calculate the answer by an interpolation between $\ln(l_{62})$ and $\ln(l_{63})$ and another interpolation between $\ln(l_{63})$ and $\ln(l_{64})$.

First, $\ln(l_{62.8}) = (1-0.8)\ln(l_{62}) + 0.8\ln(l_{63})$, which gives $l_{62.8} = 98899.79818$.

Second, $\ln(l_{63.1}) = (1-0.1)\ln(l_{63}) + 0.1\ln(l_{64})$, which gives $l_{63.1} = 98739.8354$.

Finally, $_{0.3}q_{62.8} = 1 - l_{63.1}/l_{62.8} = 0.001617.$

□

Example 2.7.

You are given the following life table:

x	l_x	d_x
90	1000	50
91	950	50
92	900	60
93	840	c_1
94	c_2	70
95	700	80

(a) Find the values of c_1 and c_2.

(b) Calculate $_{1.4}p_{90}$, assuming uniform distribution of deaths between integer ages.

(c) Repeat (b) by assuming constant force of mortality between integer ages.

Solution:

(a) We have $840 - c_1 = c_2$ and $c_2 - 70 = 700$. This gives $c_2 = 770$ and $c_1 = 70$.

(b) Assuming uniform distribution of deaths between integer ages, we have

$$
\begin{aligned}
_{1.4}p_{90} &= p_{90} \times {}_{0.4}p_{91} \\
&= p_{90}\left(1 - 0.4q_{91}\right) \\
&= \frac{l_{91}}{l_{90}}\left(1 - 0.4\frac{d_{91}}{l_{91}}\right) \\
&= \frac{950}{1000}\left(1 - 0.4 \times \frac{50}{950}\right) \\
&= 0.93.
\end{aligned}
$$

Alternatively, you can compute the answer by interpolating between l_{92} and l_{91}:

$$
_{1.4}p_{90} = p_{90} \times {}_{0.4}p_{91} = \frac{l_{91}}{l_{90}}\left(\frac{0.4l_{92} + 0.6l_{91}}{l_{91}}\right) = \frac{0.4 \times 900 + 0.6 \times 950}{1000} = 0.93
$$

(c) Assuming constant force of mortality between integer ages, we have

$$
\begin{aligned}
_{1.4}p_{90} &= p_{90} \times_{0.4} p_{91} \\
&= p_{90} \times (p_{91})^{0.4} \\
&= \frac{950}{1000}\left(\frac{900}{950}\right)^{0.4} \\
&= 0.92968.
\end{aligned}
$$

Let us conclude this section with the following table, which summarizes the formulas for the two fractional age assumptions.

	UDD	**Constant force**
$_rp_x$	$1 - rq_x$	$(p_x)^r$
$_rq_x$	rq_x	$1 - (p_x)^r$
μ_{x+r}	$\dfrac{q_x}{1 - rq_x}$	$-\ln(p_x)$

In the table, x is an integer and $0 \leq r < 1$. The shaded formulas are the key formulas that you must remember for the examination.

2.3   Select-and-Ultimate Tables

Insurance companies typically assess risk before they agree to insure you. They cannot stay in business if they sell life insurance to someone who has just discovered he has only a few months to live. A team of underwriters will usually review information about you before you are sold insurance (although there are special insurance types called "guaranteed issue" which cannot be underwritten). For this reason, a person who has just purchased life insurance has a lower probability of death than a person the same age in the general population. The probability of death for a person who has just been issued life insurance is called a select probability. In this section, we focus on the modeling of select probabilities.

Let us define the following notation.

- $[x]$ indicates the age at selection (i.e., the age at which the underwriting was done).

- $[x]+t$ indicates a person currently age $x+t$ who was selected at age x (i.e., the underwriting was done at age x). This implies that the insurance contract has elapsed for t years.

For example, we have the following select probabilities:

- $q_{[x]}$ is the probability that a life age x now dies before age $x + 1$, given that the life is selected at age x.

- $q_{[x]+t}$ is the probability that a life age $x + t$ now dies before age $x + t + 1$, given that the life was selected at age x.

Due to the effect of underwriting, a select death probability $q_{[x]+t}$ must be no greater than the corresponding ordinary death probability q_{x+t}. However, the effect of underwriting will not last forever. The period after which the effect of underwriting is completely gone is called the select period. Suppose that the select period is n years, we have

$$q_{[x]+t} < q_{x+t}, \quad \text{for } t < n.$$

$$q_{[x]+t} = q_{x+t}, \quad \text{for } t \geq n.$$

The ordinary death probability q_{x+t} is called the ultimate death probability. A life table that contains both select probabilities and ultimate probabilities is called a select-and-ultimate life table. The following is an excerpt of a (hypothetical) select-and-ultimate table with a select period of two years.

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
40	0.04	0.06	0.08	42
41	0.05	0.07	0.09	43
42	0.06	0.08	0.10	44
43	0.07	0.09	0.11	45

It is important to know how to apply such a table. Let us consider a person who is currently age 41 and is just selected. The death probabilities for this person are as follows:

Age 41: $q_{[41]} = 0.05$

Age 42: $q_{[41]+1} = 0.07$

Age 43: $q_{[41]+2} = q_{43} = 0.09$

Age 44: $q_{[41]+3} = q_{44} = 0.10$

Age 45: $q_{[41]+4} = q_{45} = 0.11$

As you see, the select-and-ultimate table is not difficult to use. We progress horizontally until we reach the ultimate death probability, then we progress vertically as when we are using an ordinary life table. To further illustrate, let us consider a person who is currently age 41 and was selected at age 40. The death probabilities for this person are as follows:

Age 41: $q_{[40]+1} = 0.06$

Age 42: $q_{[40]+2} = q_{42} = 0.08$

Age 43: $q_{[40]+3} = q_{43} = 0.09$

Age 44: $q_{[40]+4} = q_{44} = 0.10$

Age 45: $q_{[40]+5} = q_{45} = 0.11$

Even though the two persons we considered are of the same age now, their current death probabilities are different. Because the first individual has the underwriting done more recently, the effect of underwriting on him/her is stronger, which means he/she should have a lower death probability than the second individual.

We may measure the effect of underwriting by the index of selection, which is defined as follows:

$$I(x, k) = 1 - \frac{q_{[x]+k}}{q_{x+k}}.$$

For example, on the basis of the preceding table, $I(41,1) = 1 - q_{[41]+1}/q_{42} = 1 - 0.07/0.08 = 0.125$. If the effect of underwriting is strong, then $q_{[x]+k}$ would be small compared to q_{x+k}, and therefore $I(x,k)$ would be close to one. By contrast, if the effect of underwriting is weak, then $q_{[x]+k}$ would be close to q_{x+k}, and therefore $I(x,k)$ would be close to zero.

Let us first go through the following example, which involves a table of $q_{[x]}$.

Example 2.8. ⚛ **[Course 3 Fall 2001 #2]**

For a select-and-ultimate mortality table with a 3-year select period:

(i)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) White was a newly selected life on 01/01/2000.

(iii) White's age on 01/01/2001 is 61.

(iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate P.

(A) $0 \le P < 0.43$

(B) $0.43 \le P < 0.45$

(C) $0.45 \le P < 0.47$

(D) $0.47 \le P < 0.49$

(E) $0.49 \le P < 1.00$

Solution: White is now age 61 and was selected at age 60. So the probability that White will be alive 5 years from now can be expressed as $P = {}_5p_{[60]+1}$. We have

$$P = {}_5p_{[60]+1}$$
$$= p_{[60]+1} \times p_{[60]+2} \times p_{[60]+3} \times p_{[60]+4} \times p_{[60]+5}$$
$$= p_{[60]+1} \times p_{[60]+2} \times p_{63} \times p_{64} \times p_{65}$$
$$= (1 - q_{[60]+1})(1 - q_{[60]+2})(1 - q_{63})(1 - q_{64})(1 - q_{65})$$
$$= (1 - 0.11)(1 - 0.13)(1 - 0.15)(1 - 0.16)(1 - 0.17)$$
$$= 0.4589.$$

Hence, the answer is (C).

□

In some exam questions, a select-and-ultimate table may be used to model a real life problem. Take a look at the following example.

Example 2.9. 🌐 [MLC Spring 2012 #13]

Lorie's Lorries rents lavender limousines.

On January 1 of each year they purchase 30 limousines for their existing fleet; of these, 20 are new and 10 are one-year old.

Vehicles are retired according to the following 2-year select-and-ultimate table, where selection is age at purchase:

Limousine age (x)	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
0	0.100	0.167	0.333	2
1	0.100	0.333	0.500	3
2	0.150	0.400	1.000	4
3	0.250	0.750	1.000	5
4	0.500	1.000	1.000	6
5	1.000	1.000	1.000	7

Lorie's Lorries has rented lavender limousines for the past ten years and has always purchased its limousines on the above schedule.

Calculate the expected number of limousines in the Lorie's Lorries fleet immediately after the purchase of this year's limousines.

(A) 93   (B) 94   (C) 95   (D) 96   (E) 97

Solution: Let us consider a purchase of 30 limousines in a given year. According to information given, 20 of them are brand new while 10 of them are 1-year-old.

For the 20 brand new limousines, their "age at selection" is 0. As such, the sequence of "death" probabilities applicable to these 20 new limousines are $q_{[0]}$, $q_{[0]+1}$, q_2, q_3, q_4, q_5, …. Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most four years since the time of purchase. For these 20 brand new limousines, the expected number of "survivors" limousines in each future year can be calculated as follows:
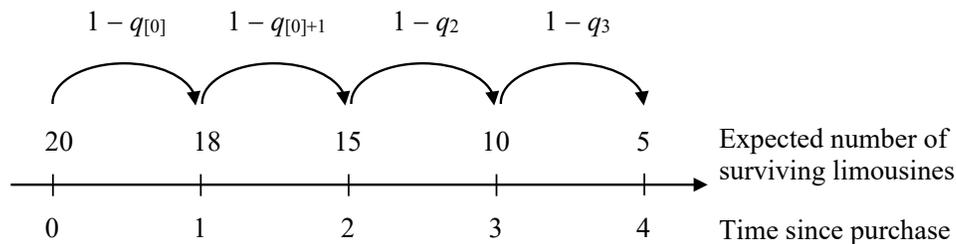
For the 10 1-year-old limousines, their "age at selection" is 1. As such, the sequence of "death" probabilities applicable to these 10 1-year-old limousines are $q_{[1]}$, $q_{[1]+1}$, q_3, q_4, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most three years since the time of purchase. For these 10 1-year-old limousines, the expected number of "surviving" limousines in each future year can be calculated as follows:
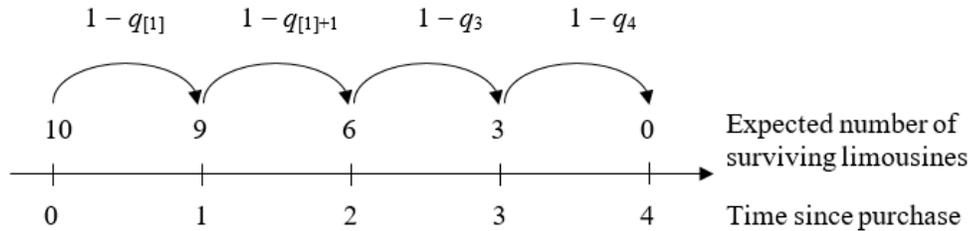
Considering the entire purchase of 30 limousines, we have the following:
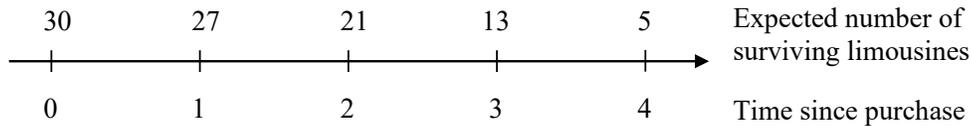
Suppose that today is January 1, 2013. Since a limousine cannot last for more than four years since the time of purchase, the oldest limousine in Lorie's fleet should be purchased on January 1, 2009. Using the results above, the expected number of limousines on January 1, 2013 can be calculated as follows:

	30	27	21	13	5
Purchased on Jan 1, 2009	30	27	21	13	5
Purchased on Jan 1, 2010		30	27	21	13
Purchased on Jan 1, 2011			30	27	21
Purchased on Jan 1, 2012				30	27
Purchased on Jan 1, 2013					30
	1/1/09	1/1/10	1/1/11	1/1/12	1/1/13
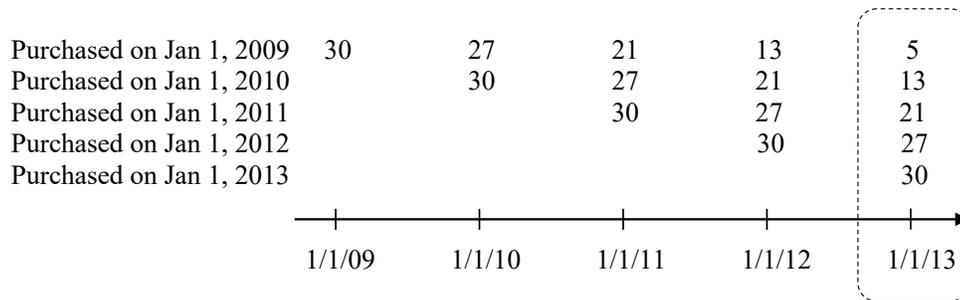
The answer is thus $5 + 13 + 21 + 27 + 30 = 96$, which corresponds to option (D).

□

Sometimes, you may be given a select-and-ultimate table that contains the life table function l_x. In this case, you can calculate survival and death probabilities by using the following equations:

$$_sp_{[x]+t} = \frac{l_{[x]+t+s}}{l_{[x]+t}}, \qquad _sq_{[x]+t} = 1 - \frac{l_{[x]+t+s}}{l_{[x]+t}}.$$

Let us study the following two examples.

Example 2.10. You are given the following select-and-ultimate table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
30	9907	9905	9901	32
31	9903	9900	9897	33
32	9899	9896	9892	34
33	9894	9891	9887	35

Calculate the following:

(a) $_2q_{[31]}$

(b) $_2p_{[30]+1}$

(c) $_{1|2}q_{[31]+1}$

Solution:

(a) $_2q_{[31]} = 1 - \dfrac{l_{[31]+2}}{l_{[31]}} = 1 - \dfrac{l_{33}}{l_{[31]}} = 1 - \dfrac{9897}{9903} = 0.000606.$

(b) $_2p_{[30]+1} = \dfrac{l_{[30]+1+2}}{l_{[30]+1}} = \dfrac{l_{33}}{l_{[30]+1}} = \dfrac{9897}{9905} = 0.999192.$

(c) $_{1|2}q_{[31]+1} = \dfrac{l_{[31]+1+1} - l_{[31]+1+1+2}}{l_{[31]+1}} = \dfrac{l_{33} - l_{35}}{l_{[31]+1}} = \dfrac{9897 - 9887}{9900} = 0.001010.$ □

Exam questions such as the following may involve both $q_{[x]}$ and $l_{[x]}$.

Example 2.11. **[MLC Spring 2012 #1]**

For a 2-year select and ultimate mortality model, you are given:

(i) $q_{[x]+1} = 0.95q_{x+1}$

(ii) $l_{76} = 98{,}153$

(iii) $l_{77} = 96{,}124$

Calculate $l_{[75]+1}$.

(A) 96,150   (B) 96,780   (C) 97,420   (D) 98,050   (E) 98,690

Solution: From (ii) and (iii), we know that $q_{76} = 1 - 96124/98153 = 0.020672.$

From (i), we know that $q_{[75]+1} = 0.95q_{76} = 0.95 \times 0.020672 = 0.019638.$

Since
$$l_{[75]+2} = l_{[75]+1}(1 - q_{[75]+1}),$$
and $l_{[75]+2} = l_{77}$, we have $l_{[75]+1} = 96124/(1 - 0.019638) = 98049.5.$ The answer is (D).   □

Mock Test 1

1. For a whole life insurance of 1,000 issued to life selected at age x,

   (i) Percent of premium expenses is 90% in the first year, and 10% in each year thereafter.

   (ii) Maintenance expenses are 15 per 1,000 of insurance in the first year, and 3 per 1,000 of insurance thereafter.

   (iii) Claim settlement expenses are 10 per 1,000 of insurance.

   (iv) A 15-year select and ultimate mortality is to be used.

   Determine the expression of the gross premium for the policy using equivalence principle.

   (A) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (B) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

   (C) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{\ddot{a}_{[x]} - 0.9}$

   (D) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (E) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

2. The following results were obtained from a survival study, using the Kaplan-Meier (KM) estimator:

Time of death t in the sample	KM estimate of $S(t)$	Standard error of estimate
17	0.957	0.0149
25	0.888	0.0236
32	0.814	0.0298
36	0.777	0.0321
39	0.729	0.0348
42	0.680	0.0370
44	0.659	0.0378
47	0.558	0.0418
50	0.360	0.0470
54	0.293	0.0456
56	0.244	0.0440
57	0.187	0.0420
59	0.156	0.0404
62	0.052	0.0444

Find the upper limit of the 95% log-confidence interval for $S(45)$.

   (A) 0.70   (B) 0.71   (C) 0.72   (D) 0.73   (E) 0.74

---

3. :: You are given:

(i) $i = 0.07$

(ii) $\ddot{a}_x = 11.7089$

(iii) $\ddot{a}_{x:\overline{40}|} = 11.55$

(iv) $\ddot{a}_{x+40} = 7.1889$

Find $A^1_{x:\overline{40}|}$.

(A) 0.120   (B) 0.147   (C) 0.172   (D) 0.197   (E) 0.222

4. :: You are given the following select-and-ultimate table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.11	0.13	0.15	67
66	0.12	0.135	0.16	68
67	0.13	0.145	0.17	69

Deaths are uniformly distributed over each year of age.

Find $_{1.8}p_{[66]+0.6}$.

(A) 0.69   (B) 0.71   (C) 0.73   (D) 0.75   (E) 0.77

5. :: For a 3-year fully discrete endowment insurance of 1,000 on (x), you are given:

(i) $q_x = 0.1$

(ii) $q_{x+1} = 0.15$

(iii) $v = 0.9$

(iv) Deaths are uniformly distributed over each year of age.

Calculate the net premium policy value 9 months after the issuance of the policy.

(A) 274   (B) 280   (C) 283   (D) 286   (E) 290

6. :: Which of the following statements is/are correct?

I. Insurable interest in an entity exists if one would suffer a financial loss if that entity is damaged.

II. Insurable interest is related to the concept of adverse selection.

III. Stranger owned life insurance is illegal in many jurisdiction because the purchaser has no insurable interest in the insured.

(A) I only   (B) II only   (C) I and III only

(D) II and III only   (E) I, II and III

7. Which of the following is/are strictly increasing function(s) of T_x for all $T_x \geq 0$?

   I. The present value random variable for a continuous whole life annuity of \$1 on (x)

   II. The present value random variable for a continuous n-year temporary life annuity of \$1 on (x)

   III. The net future loss at issue random variable for a fully continuous whole life insurance of \$1 on (x)

   (A) I only   (B) III only   (C) I, II only

   (D) I, III only   (E) I, II and III

8. For a fully discrete whole life insurance on (30), you are given:

   (i) $i = 0.05$

   (ii) $q_{29+h} = 0.004$

   (iii) The net amount at risk for policy year h is 1295.

   (iv) The terminal policy value for policy year $h - 1$ is 179.

   (v) $\ddot{a}_{30} = 16.2$

   Calculate the initial policy value for policy year $h + 1$.

   (A) 188   (B) 192   (C) 200   (D) 214   (E) 226

9. You are given:

   (i) $\mu_{x+t} = \begin{cases} 0.02 & 0 \leq t < 1 \\ 0.07 & 1 \leq t < 2 \end{cases}$ (ii) $Y = \min(T_x, 2)$

   Calculate $E(Y)$.

   (A) 1.88   (B) 1.90   (C) 1.92   (D) 1.94   (E) 1.96

10. An insurer issues fully discrete endowment insurance policies to a group of 50 high-risk drivers all aged 35 with a sum insured of 2,000. You are given:

    (i) The mortality of each driver follows the Standard Ultimate Life Table with an age rating of 3 years. That is,
    $$q_x = q_{x+3}^{SULT},$$
    where q_y^{SULT} is the 1-year death probability under the Standard Ultimate Life Table.

    (ii) Lifetimes of the group of 50 high-risk drivers are independent.

    (iii) Premiums are payable annually in advance. Each premium is 110% of the net annual premium.

    (iv) $i = 0.05$

    By assuming a normal approximation, estimate the 95th-percentile of the net future loss random variable.

    (A) 1090   (B) 1240   (C) 1390   (D) 1540   (E) 1690

11. $\bullet$ For a population, the force of mortality is $\mu_x = 0.01\pi \tan(0.01\pi x)$ for $0 < x < 50$.

Calculate $_{10|5}q_{20}$.

[Note: The anti-derivative of $\tan x$ is $-\ln|\cos x|$.]

    (A) 0.15   (B) 0.17   (C) 0.19   (D) 0.21   (E) 0.23

12. $\bullet$ Which of the following regarding is/are correct?

   I. Variable annuity has cash value.

  II. Term insurance has cash value.

 III. Universal life insurance has cash value.

    (A) I only   (B) II only   (C) III only

    (D) II and III only   (E) I, II and III

13. $\bullet$ You are given the following about 100 insurance policies in a study of time to policy surrender:

  (i) The study was designed in such a way that for every policy that was surrendered, a new policy was added, meaning that the risk set, r_j, is always equal to 100.

  (ii) Policies are surrendered only at the end of a policy year.

 (iii) The number of policies surrendered at the end of each policy year was observed to be:

$$2 \text{ at the end of the } 1^{\text{st}} \text{ policy year}$$
$$3 \text{ at the end of the } 2^{\text{nd}} \text{ policy year}$$
$$4 \text{ at the end of the } 3^{\text{rd}} \text{ policy year}$$
$$\vdots$$
$$n + 1 \text{ at the end of the } n^{\text{th}} \text{ policy year}$$

 (iv) The Nelson-Aalen empirical estimate of the cumulative distribution function at time n is 0.41725.

Find n.

    (A) 9   (B) 10   (C) 11   (D) 12   (E) 13

14. You are given:

    (i) $\delta = 0.05$

    (ii) $\bar{A}_x = 0.44$

    (iii) $^2\bar{A}_x = 0.22$

    Consider a portfolio of 100 fully continuous whole life insurances. The ages of the all insureds are x, and their lifetimes are independent. The face amount of the policies, the premium rate and the number of policies are as follows:

Face amount	Premium rate	Number of Policies
100	4.3	75
400	17.5	25

    By using a normal approximation, calculate the probability that the present value of the aggregate loss-at-issue for the insurer's portfolio will exceed 700.

    (A) $1 - \Phi(2.28)$ (B) $1 - \Phi(0.17)$ (C) $\Phi(0)$

    (D) $\Phi(0.17)$ (E) $\Phi(2.28)$

15. Victor is now age 22, and his future lifetime has the following cumulative distribution function:

    $$\mathrm{F}_{22}(t) = 1 - (1 + 0.04t)e^{-0.04t}.$$

    Let Z be the present value random variable for a fully continuous life insurance that pays 100 immediately on the death of Victor provided that he dies between ages 32 and 52.

    The force of interest is 0.06.

    Find the 70th-percentile of Z.

    (A) 0   (B) 25   (C) 40   (D) 47   (E) 50

16. You are given:

    (i) $i = 0.05$

    (ii) $A_{60} = 0.560$

    (iii) $A_{40} = 0.176$

    (iv) $_{20}p_{40} = 0.75$

    Calculate $a^{(3)}_{40:\overline{20}|}$ using the two-term Woolhouse's approximation.

    (A) 13.7   (B) 14.2   (C) 14.5   (D) 14.9   (E) 15.1

17. ⁖ You are given:

   (i) $\delta = 0.05$

   (ii) $\ddot{a}_{60} = 12.18$

   (iii) $p_{60} = 0.98$

Using the claims accelerated approach, calculate $\ddot{a}_{61}^{(6)}$.

   (A) 11.6   (B) 11.7   (C) 11.8   (D) 11.9   (E) 12.1

18. ⁖ Let Y be the present value random variable for a special three-year temporary life annuity on (x). You are given:

   (i) The life annuity pays $2 + k$ at time k, for $k = 0$, 1 and 2.

   (ii) $v = 0.9$

   (iii) $p_x = 0.8$, $p_{x+1} = 0.75$, $p_{x+2} = 0.5$

Calculate the standard deviation of Y.

   (A) 1.2   (B) 1.8   (C) 2.4   (D) 3.0   (E) 3.6

19. ⁖ You are given:

$$\mu_x = \begin{cases} 0.04 & 50 \le x < 60 \\ 0.05 + 0.001(x - 60)^2 & 60 \le x < 70 \end{cases}$$

Calculate $_{4|14}q_{50}$.

   (A) 0.38   (B) 0.44   (C) 0.47   (D) 0.50   (E) 0.56

20. ⁖ You are given:

   (i) $p_{41} = 0.999422$

   (ii) $i = 0.03$

   (iii) $\ddot{a}_{42:\overline{23|}} = 16.7147$

Calculate the Full Preliminary Term reserve at time 2 for a 25-year fully discrete endowment insurance, issued to (40), with sum insured 75,000.

   (A) 2,188   (B) 2,190   (C) 2,192   (D) 2,194   (E) 2,196

** END OF EXAMINATION **

1. [Chapter 7] Answer: (B)

   Let G be the gross premium. By the equivalence principle,

   $$\text{APV of gross premiums} = \text{APV of death benefit} + \text{APV of expenses}.$$

   $$
   \begin{aligned}
   G\ddot{a}_{[x]} &= 1010\bar{A}_{[x]} + 0.9G + 0.1Ga_{[x]} + 15 + 3a_{[x]} \\
   &= 1010\bar{A}_{[x]} + 0.9G + 0.1G(\ddot{a}_{[x]} - 1) + 15 + 3(\ddot{a}_{[x]} - 1) \\
   G &= \frac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}
   \end{aligned}
   $$

2. [Chapter 8] Answer: (D)

   $$U = \frac{1.96\text{SE}[\hat{S}(t)]}{\hat{S}(t)\ln\hat{S}(t)} = \frac{1.96 \times 0.0378}{0.659\ln 0.659} = -0.26958, \text{ and the upper limit is}$$

   $0.659^{\exp(-0.26958)} = 0.659^{0.763697} = 0.727249.$

3. [Chapter 4] Answer: (E)

   We first change all annuities into insurances.

   Statement (ii) implies that $A_x = 1 - \frac{0.07}{1.07} \times 11.7089 = 0.2340.$

   Statement (iii) implies that $A_{x:\overline{40|}} = 1 - \frac{0.07}{1.07} \times 11.55 = 0.2444.$

   Statement (iv) implies that $A_{x+40} = 1 - \frac{0.07}{1.07} \times 7.1889 = 0.5297.$

   Finally, by $A_x = A^1_{x:\overline{40|}} + A_{x:\overline{40|}}^{1}A_{x+40}$ and $A_{x:\overline{40|}}^{1} + A^1_{x:\overline{40|}} = A_{x:\overline{40|}}$,

   $$0.2340 = A^1_{x:\overline{40|}} + (0.2444 - A^1_{x:\overline{40|}}) \times 0.5297$$

   On solving, we get $A^1_{x:\overline{40|}} = \dfrac{0.2340 - 0.2444 \times 0.5297}{1 - 0.5297} = 0.2223.$

4. [Chapter 2] Answer: (E)

   Method 1:
   $$
   \begin{aligned}
   {}_{1.8}p_{[66]+0.6} &= {}_{0.4}p_{[66]+0.6} \times {}_{1.4}p_{[66]+1} \\
   &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{[66]+2} \\
   &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{68}
   \end{aligned}
   $$

   Obviously, $p_{[66]+1} = 1 - 0.135 = 0.865$, ${}_{0.4}p_{68} = 1 - 0.4q_{68} = 1 - 0.4(0.16) = 0.936.$

   Finally, ${}_{0.4}p_{[66]+0.6} = p_{[66]}/{}_{0.6}p_{[66]} = 0.88/(1 - 0.6 \times 0.12) = 0.948276.$

   So, the answer is ${}_{1.8}p_{[66]+0.6} = 0.76776.$

   Method 2:
   $${}_{1.8}p_{[66]+0.6} = l_{[66]+2.4}/l_{[66]+0.6}$$

   Without loss of generality, let $l_{[66]} = 100$. Then we have:

   $l_{[66]+1} = 88$, $l_{[66]+2} = 88 \times 0.865 = 76.12$, $l_{[66]+2} = 76.12 \times 0.84 = 63.9408.$

   So, $l_{[66]+2.4} = 0.4 \times 63.9408 + 0.6 \times 76.12 = 71.24832,$

   $l_{[66]+0.4} = 0.6 \times 88 + 0.4 \times 100 = 92.8$, and ${}_{1.8}p_{[66]+0.6} = 0.767762.$

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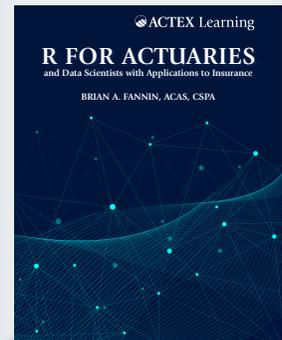
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